The exact evaluation of the corner-to-corner resistance of an $M \times N$ resistor network:

asymptotic expansion

# The exact evaluation of the corner-to-corner resistance of an $M \times N$ resistor network: asymptotic expansion

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**Abstract**
We study the corner-to-corner resistance of an $M \times N$ resistor network with
resistors $r$ and $s$ in the two spatial directions and obtain an asymptotic expansion
of its exact expression for large $M$ and $N$. For $M = N$, $r = s = 1$, our result is

$$R_{N \times N} = \frac{4}{\pi} \log N + 0.077\,318 + \frac{0.266\,070}{N^2} - \frac{0.534\,779}{N^4} + O\left(\frac{1}{N^6}\right).$$

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## 1. Introduction

A classic problem in the theory of electric circuits is the computation of the resistance between
two nodes in a resistor network. Formulated by Kirchhoff [1] more than 160 years ago, the
problem has been studied by numerous authors over many years (see, for example, [2, 3]).
Kirchhoff explored the graph-theoretical aspect of the algebraic formulation and obtained the
two-point resistance in terms of two-rooted spanning forests and spanning trees. But the
formulation, while elegant, does not provide sufficient physical insights. Past studies have
instead focused on infinite networks for which analysis can be carried to fruition [4].

The computation of the asymptotic expansion of the corner-to-corner resistance of a
rectangular resistor network has been of interest for some time, as its value provides a lower
bound to the resistance of compact percolation clusters in the Domany–Kinzel model of a
directed percolation [5]. The corner-to-corner resistance has been studied by one of us (JWE)
numerically using the method of a differential approximant [6] together with a Neville table
analysis [7].

Recently, one of us (FYW) has revisited the two-point resistance problem [8] and deduced
a closed-form expression for the resistance between arbitrary two nodes for finite networks.
However, the exact expression obtained in [8] is in the form of a double summation whose
mathematical and physical contents are not immediately apparent. In this paper, we take a
closer look at this summation formula and obtain its asymptotic expansion for large lattices.
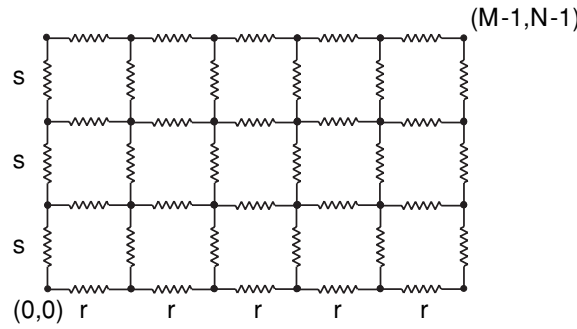
**Figure 1.** An $M \times N$ resistor network.

The organization of this paper is as follows. In section 2 we recall the expression of the corner-to-corner resistance in an $M \times N$ resistor network obtained in [8] and reduce it to a form more manageable for our purposes. One of the two summations in the resistance expression is carried out in section 3 by using a summation identity which we derive. The resulting expression is written in the form of a dominant term plus a correction. Asymptotic expansions of the dominant and correction terms are obtained in sections 4 and 5, and we summarize the results in section 6. We also show that the exact expression of the asymptotic expansion is in agreement with those determined numerically [7].

## 2. Formulation of the summation formula

Consider a rectangular $M \times N$ network of resistors with resistances $r$ and $s$ on edges of the network in the respective horizontal and vertical directions. For definiteness, we consider both $M$, $N$ even, and expect the asymptotic expansion to be independent of this choice. The example of an $M = 6$, $N = 4$ network is shown in figure 1.

Using equation (37) of [8], the resistance between opposite corner nodes $(0, 0)$ and $(M - 1, N - 1)$ of the network is

$$R_{\{M \times N\}}(r, s) = \frac{r(M - 1)}{N} + \frac{s(N - 1)}{M}$$
$$+ \frac{2}{MN} \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} \frac{\left[ \cos\left(\frac{1}{2}\theta_m\right) \cos\left(\frac{1}{2}\phi_n\right) - \cos\left(M - \frac{1}{2}\right)\theta_m \cos\left(N - \frac{1}{2}\right)\phi_n \right]^2}{r^{-1}(1 - \cos\theta_m) + s^{-1}(1 - \cos\phi_n)},$$

(1)

where $\theta_m = m\pi/M$, $\phi_n = n\pi/N$. Rearranging the numerator in the summand, (1) becomes

$$R_{M \times N}(r, s) = \frac{r(M - 1)}{N} + \frac{s(N - 1)}{M}$$
$$+ \frac{8}{MN} \sum_{m=1}^{M-1} \sum_{\substack{n=1 \\ (m+n \text{ odd})}}^{N-1} \frac{\cos^2(\theta_m/2) \cos^2(\phi_n/2)}{r^{-1}(1 - \cos\theta_m) + s^{-1}(1 - \cos\phi_n)}.$$

(2)

There are two possibilities for the restriction $m + n = $ odd to hold, namely,

$$m = 2p - 1, n = 2q, \qquad p = 1, 2, \ldots, M/2, \qquad q = 1, 2, \ldots, N/2,$$
$$n = 2p - 1, m = 2q, \qquad p = 1, 2, \ldots, N/2, \qquad q = 1, 2, \ldots, M/2.$$

Splitting the sum into two parts accordingly and introducing the notation

$$A_q = \frac{q\pi}{N}, \qquad B_p = \left(p - \frac{1}{2}\right)\frac{\pi}{M},$$

we obtain

$$R_{M\times N}(r, s) = (rs)^{\frac{1}{2}}[R_{M\times N}(r/s) + R_{N\times M}(s/r)], \qquad (3)$$

where

$$R_{M\times N}(\rho) = \frac{\sqrt{\rho}(M-1)}{N} + \frac{4\sqrt{\rho}}{MN}\sum_{p=1}^{M/2}\sum_{q=1}^{N/2}\left[\frac{\cos^2 A_q(1 + \rho\sin^2 A_q)}{\rho\sin^2 A_q + \sin^2 B_p} - \cos^2 A_q\right]. \qquad (4)$$

Sums of the term $\cos^2 A_q$ can be carried out using the identity

$$\sum_{q=1}^{N/2}\cos^2\left(\frac{q\pi}{N}\right) = \frac{N}{4} - \frac{1}{2}. \qquad (5)$$

This yields

$$R_{M\times N}(\rho) = \sqrt{\rho}\left(\frac{M}{N} - \frac{1}{2}\right) + S_{M\times N}(\rho)$$

and

$$R_{M\times N}(r, s) = \sqrt{rs}\left[\sqrt{\rho}\left(\frac{M}{N} - \frac{1}{2}\right) + \frac{1}{\sqrt{\rho}}\left(\frac{N}{M} - \frac{1}{2}\right) + S_{M\times N}(\rho) + S_{N\times M}(1/\rho)\right], \qquad (6)$$

where

$$S_{M\times N}(\rho) = \frac{4\sqrt{\rho}}{N}\sum_{q=1}^{N/2}(\cos^2 A_q)(1 + \rho\sin^2 A_q)S_{q,M,N}(\rho) \qquad (7)$$

with

$$S_{q,M,N}(\rho) = \frac{1}{M}\sum_{p=1}^{M/2}[\rho\sin^2 A_q + \sin^2 B_p]^{-1}$$

$$= \frac{1}{M}\sum_{k=0}^{(M/2)-1}\left[\rho\sin^2 A_q + \sin^2\left(\frac{(k+\frac{1}{2})\pi}{M}\right)\right]^{-1}. \qquad (8)$$

## 3. Evaluation of $S_{q,M,N}(\rho)$

It is tempting to evaluate the summation (8) by using the Euler–Maclaurin summation formula. But as shown in the appendix the Euler–Maclaurin summation is inadequate since it does not determine an error term which cannot be ignored. We proceed here to evaluate $S_{q,M,N}(\rho)$ by using a summation identity which we state as a lemma:

**Lemma.**

$$\sum_{k=0}^{(M/2)-1}\frac{1}{\rho\sin^2 A_q + \sin^2\left[(k+\frac{1}{2})\frac{\pi}{M}\right]} = R(y^*) \equiv \frac{M\tanh(\pi y^*)}{2\sqrt{\rho}\sin A_q\sqrt{1 + \rho\sin^2 A_q}}, \qquad (9)$$

where $M = even$ and $y^* = y^*_{q,M,N}(\rho)$ is defined by

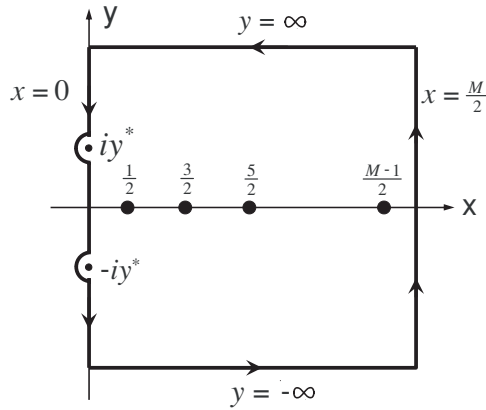$$\sinh\frac{\pi y^*}{M} = \sqrt{\rho}\sin A_q. \qquad (10)$$

**Figure 2.** Contour of integration $C$ in (11). Solid circles denote simple poles enclosed by $C$.

**Proof.** Consider the contour integral

$$J_{q,M,N}(\rho) = \frac{1}{2\pi i} \oint_C \frac{\pi \tan(\pi z)\, dz}{\sin^2\left(\frac{\pi z}{M}\right) + \rho \sin^2 A_q}, \tag{11}$$

where the contour $C$ consists of the lines

$$x = \frac{M}{2}, \qquad y = -\infty, \qquad y = \infty \tag{12}$$

and the imaginary axis $x = 0$ with two half circles of radii $\epsilon \to 0$ around the two points $z = \pm iy^*$ as shown in figure 2. The contour encloses $\frac{M}{2} + 2$ simple poles of the integrand at $z = \pm iy^*$ and $z = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{M-1}{2}$. The residue is $R(y^*)$ at the simple poles on the $y$-axis and $-\left[\rho \sin^2 A_q + \sin^2\left(k + \frac{1}{2}\right)\frac{\pi}{M}\right]^{-1}$ at $z = k + \frac{1}{2}, k = 0, 1, \ldots$.

The integration along the contour $C$ vanishes on the lines $y = \pm\infty$ and on the straight line portions of $x = 0, \frac{M}{2}$ since the integrand is odd in $y$. Hence the contour integral is nonzero only on the two half circles. The integrand is odd in $z$ so that the integral along the lower half circle is equal to the integral in the anti-clockwise direction along the reflection of the upper half circle in the $y$-axis. The integral $J_{q,M,N}(\rho)$ along the contour $C$ may therefore be obtained by integrating round a circle centered on $iy^*$. Thus, by the residue theorem, the residue at $iy^*$ is equal to the sum of the residues of the $\frac{M}{2} + 2$ simple poles enclosed by $C$, hence

$$R(y^*) = 2R(y^*) - \sum_{k=0}^{(M/2)-1} \frac{1}{\sin^2\left[\left(k + \frac{1}{2}\right)\frac{\pi}{M}\right] + \rho \sin^2 A_q}, \tag{13}$$

which yields (9).                                                                                       □

The substitution of (9) into (8) and (7) now yields

$$S_{q,M,N}(\rho) = \frac{\tanh(\pi y^*)}{2\sqrt{\rho}\, \sin A_q \sqrt{1 + \rho \sin^2 A_q}} \tag{14}$$

and

$$S_{M \times N}(\rho) = \sum_{q=1}^{N/2} D_{q,N}(\rho) \tanh(\pi y^*), \tag{15}$$

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where

$$D_{q,N}(\rho) = \frac{2}{N} \cdot \frac{\cos^2 A_q \sqrt{1 + \rho \sin^2 A_q}}{\sin A_q}. \tag{16}$$

Anticipating that the dominate contribution of $S_{M \times N}(\rho)$ is given by (15) with $\tanh(\pi y^*)$ replaced by 1 (see the appendix), we rewrite (15) as

$$S_{M \times N}(\rho) = S_N^{(1)}(\rho) + \Delta_{M,N}(\rho), \tag{17}$$

where

$$S_N^{(1)}(\rho) = \sum_{q=1}^{N/2} D_{q,N}(\rho) \tag{18}$$

is the dominate contribution, and

$$\Delta_{M,N}(\rho) = \sum_{q=1}^{N/2} \Delta_{q,M,N}(\rho) \tag{19}$$

is the correction with

$$\Delta_{q,M,N}(\rho) = D_{q,N}(\rho)[\tanh(\pi y^*) - 1]. \tag{20}$$

Numerical evaluation of the difference $\Delta_{q,M,N}(1)$ using $\tanh(\pi y^*)$ given by (9) for $M = N$ and small values of $q$ shows that it initially decreases with $N$ but ultimately shows a rapid increase. For $q = 1$ the turning point is $N = 6$ and for $q = 2$ it is $N = 12$. However $\Delta_{q,M,N}(\rho)$ for fixed $N$ decreases exponentially with increasing $q$, a fact which will be seen to hold for general $M$ and $N$ later (see equation (38)). The sum in (19) therefore converges rapidly.

The two terms $S_N^{(1)}(\rho)$ and $\Delta_{M,N}(\rho)$ in (17) are evaluated in the following two sections.

## 4. Evaluation of $S_N^{(1)}(\rho)$

The asymptotic form of $S_N^{(1)}(\rho)$ given by the summation (18) is now deduced using the Euler–Maclaurin sum formula ([9] equation (5.8.13)) and Bernoulli numbers $B_{2i}$

$$\sum_{p=1}^{r} f_p = \frac{1}{h} \int_{x_0}^{x_r} f(x) \, dx + \frac{1}{2}[f(x_r) - f(x_0)]$$

$$+ \sum_{i=1}^{m} \frac{B_{2i} h^{2i-1}}{(2i)!} [f^{(2i-1)}(x_r) - f^{(2i-1)}(x_0)] + E_m(\eta_m), \tag{21}$$

where $f_p$ is such that $f_p = f(x_0 + ph)$, the integer $r$ is finite and the error term is given by

$$E_m(\eta_m) = r \frac{B_{2m+2} h^{2m+2}}{(2m+2)!} f^{(2m+2)}(\eta_m), \qquad x_0 < \eta_m < x_r. \tag{22}$$

But the direct application of (21) to effect the summation in (18) leads to a divergent integral so we add and subtract $1/A_q$ to the summand and use (21) with $f(x)$ given by

$$f(x) \equiv f_\rho(x) = \frac{\cos^2 x}{\sin x} \sqrt{1 + \rho \sin^2 x} - \frac{1}{x}. \tag{23}$$

Using $x_0 = 0$, $x_r = \pi/2$, $h = \pi/N$, $r = N/2$ and since $f_\rho(x)$ does not diverge at small $x$, the error term $E_m$ is of the order of $O(N^{-(2m+1)})$ and can be neglected in $m \to \infty$. Denoting by

$U_N(\rho)$ and $L_N(\rho)$ the respective correction to the integral at the upper and lower limits, we obtain

$$S_N^{(1)}(\rho) = I(\rho) + S_N + U_N(\rho) + L_N(\rho), \tag{24}$$

where $I(\rho)$ is the integral

$$
\begin{aligned}
I(\rho) &= \frac{2}{\pi} \int_0^{\pi/2} f_\rho(x)\, \mathrm{d}x \\
&= \frac{1}{\pi}\left[ -1 + 4\log 2 - 2\log\pi - \log(1+\rho) + \frac{\rho-1}{\sqrt{\rho}} \tan^{-1}\sqrt{\rho} \right]. \tag{25}
\end{aligned}
$$

The second term in (24) is the added summation $S_N$, which can be evaluated using the result ([9] chapter 5, problem 26) as

$$S_N = \frac{2}{N}\sum_{q=1}^{N/2}\frac{1}{A_q} = \frac{2}{\pi}\sum_{q=1}^{N/2}\frac{1}{q} = \frac{2}{\pi}\left( \log\frac{N}{2} + \gamma + \frac{1}{N} - \sum_{m=1}^{\infty}\frac{4^m B_{2m}}{2mN^{2m}} \right), \tag{26}$$

where $\gamma = 0.577\,215\,664\,901\,53\ldots$ is Euler's constant.

The first part of $f_\rho(x)$ is antisymmetric about $\pi/2$ so the odd derivatives at the upper limit arise entirely from the $-1/x$ term and is independent of $\rho$. Hence for $j$ odd $f_\rho^{(j)}(\pi/2) = (-1)^{j+1} j! (2/\pi)^{j+1}$ and the correction to the integral from the upper limit is

$$U_N(\rho) = \frac{1}{N}f\left(\frac{\pi}{2}\right) + \frac{2}{N}\sum_{i=1}^{m}\frac{B_{2i}h^{2i-1}}{(2i)!}f_\rho^{(2i-1)}\left(\frac{\pi}{2}\right) = \frac{-2}{\pi N} + \frac{2}{\pi}\sum_{i=1}^{m}\frac{4^i B_{2i}}{2iN^{2i}}, \tag{27}$$

which, as $m \to \infty$, cancels the terms of the inverse powers of $N$ in $S_N$.

At the lower limit we have $f_\rho(0) = 0$ and

$$L_N(\rho) = -\frac{2}{\pi}\sum_{i=1}^{m}\frac{B_{2i}}{(2i)!}\left(\frac{\pi}{N}\right)^{2i}f_\rho^{(2i-1)}(0). \tag{28}$$

Using Bernoulli numbers $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$ ([9] equation (5.8.8)), the leading terms in $L_N$ are

$$L_N(\rho) = \frac{2}{\pi}\left[ -\frac{\pi^2}{12N^2}f_\rho^{(1)}(0) + \frac{\pi^4}{720N^4}f_\rho^{(3)}(0) - \frac{\pi^6}{30240N^6}f_\rho^{(5)}(0) + O\left(\frac{1}{N^8}\right) \right] \tag{29}$$

with

$$
\begin{aligned}
f_\rho^{(1)}(0) &= \tfrac{1}{6}(-5 + 3\rho), \\
f_\rho^{(3)}(0) &= \tfrac{1}{60}(67 - 210\rho - 45\rho^2), \\
f_\rho^{(5)}(0) &= \tfrac{1}{126}(-95 + 3843\rho + 2835\rho^2 + 945\rho^3).
\end{aligned} \tag{30}
$$

Combining (24)–(27), we obtain the result

$$S_N^{(1)}(\rho) = I(\rho) + \frac{2}{\pi}\left[ \log\frac{N}{2} + \gamma \right] + L_N(\rho). \tag{31}$$

## 5. Evaluation of $\Delta_{q,M,N}(\rho)$

We now evaluate $\Delta_{M,N}(\rho)$ given by the summation (19) with $\Delta_{q,M,N}(\rho)$ given by (20).

6

For large $M$, $N$ with $M/N = \lambda$ fixed, we use

$$\sinh^{-1}(\sqrt{\rho}\sin x) = \sqrt{\rho}x\left[1 - \frac{1+\rho}{6}x^2 + \frac{(1+\rho)(1+9\rho)}{120}x^4 + \cdots\right] \qquad (32)$$

and (10) to obtain

$$\pi y^* = (\pi\tilde{q})\left[1 - \frac{1+\rho}{6}\left(\frac{q\pi}{N}\right)^2 + \frac{(1+\rho)(1+9\rho)}{120}\left(\frac{q\pi}{N}\right)^4 + \cdots\right], \qquad (33)$$

where $\tilde{q} = \lambda\sqrt{\rho}q$. This leads to

$$\tanh(\pi y^*) = \tanh(\pi\tilde{q}) - \frac{1+\rho}{6}(\pi\tilde{q})\operatorname{sech}^2(\pi\tilde{q})\left(\frac{\pi q}{N}\right)^2 + \left[\frac{\pi\tilde{q}}{120}(1+\rho)(1+9\rho)\right.$$
$$\left. - \frac{(\pi\tilde{q})^2}{36}(1+\rho)^2\tanh(\pi\tilde{q})\right]\operatorname{sech}^2(\pi\tilde{q})\left(\frac{\pi q}{N}\right)^4 + \cdots. \qquad (34)$$

Substituting (34) into (20), we obtain

$$\Delta_{q,M,N}(\rho) = D_{q,N}(\rho)[\tanh(\pi\tilde{q}) - 1] + D_{q,N}(\rho) \times (\pi\tilde{q})\left\{-\frac{1+\rho}{6}\operatorname{sech}^2(\pi\tilde{q})\left(\frac{\pi q}{N}\right)^2\right.$$
$$\left. + (1+\rho)\operatorname{sech}^2(\pi\tilde{q})\left[\frac{1+9\rho}{120} - \frac{1+\rho}{36}(\pi\tilde{q})\tanh(\pi\tilde{q})\right]\left(\frac{\pi q}{N}\right)^4 + \cdots\right\}. \qquad (35)$$

Rewrite $D_{q,N}(\rho)$ given by (16) as

$$D_{q,N}(\rho) = \frac{2}{q\pi} + \frac{2}{N}f_\rho\left(\frac{\pi q}{N}\right)$$
$$= \frac{1}{q\pi}\left[2 + 2f_\rho^{(1)}(0)\left(\frac{q\pi}{N}\right)^2 + \frac{1}{3}f_\rho^{(3)}(0)\left(\frac{q\pi}{N}\right)^4 + \cdots\right], \qquad (36)$$

where the derivatives are given in (30). This leads to the desired asymptotic expansion

$$\Delta_{q,M,N}(\rho) = \sum_{i=0}^{\infty}\frac{\Delta_{q,2i}(\lambda,\rho)}{N^{2i}} \qquad (37)$$

with expansion coefficients

$$\Delta_{q,0}(\lambda,\rho) = \frac{2}{\pi q}[\tanh(\pi\tilde{q}) - 1],$$

$$\Delta_{q,2}(\lambda,\rho) = 2\pi q f_\rho^{(1)}(0)[\tanh(\pi\tilde{q}) - 1] - \frac{\lambda\sqrt{\rho}(\pi q)^2}{3}(1+\rho)\operatorname{sech}^2(\pi\tilde{q}),$$

$$\Delta_{q,4}(\lambda,\rho) = \frac{(\pi q)^3 f_\rho^{(3)}(0)}{3}[\tanh(\pi\tilde{q}) - 1]$$
$$+ \lambda\sqrt{\rho}(\pi q)^4(1+\rho)\left[\frac{53-3\rho}{180} - \frac{(1+\rho)}{18}(\pi\tilde{q})\tanh(\pi\tilde{q})\right]\operatorname{sech}^2(\pi\tilde{q}). \qquad (38)$$

As remarked earlier, values of these coefficients decrease exponentially as $q$ increases.

**Table 1.** The coefficients $\Delta_{q,2i}(1, 1)$ in (43).

| $q$ | $2\Delta_{q,0}$ | $2\Delta_{q,2}$ | $2\Delta_{q,4}$ |
|---|---|---|---|
| 1 | $-0.004\,746\,746\,975\,499\,7281$ | $-0.082\,316\,316\,898\,659\,221$ | $0.038\,515\,515\,969\,807\,909$ |
| 2 | $-4.440\,206\,206\,434\,262810^{-6}$ | $-0.000\,675\,675\,475\,810\,569\,74$ | $-0.032\,940\,940\,383\,097\,552$ |
| 3 | $-5.527\,907\,907\,846\,738\,310^{-9}$ | $-2.921\,521\,521\,929\,085\,010^{-6}$ | $-0.000\,609\,609\,399\,827\,443\,05$ |
| 4 | $-7.742\,287\,287\,885\,427\,210^{-12}$ | $-9.835\,001\,001\,654\,764\,310^{-9}$ | $-5.326\,415\,415\,423\,713\,010^{-6}$ |
| 5 | $-1.156\,662\,662\,112\,178\,110^{-14}$ | $-2.893\,517\,517\,142\,470\,410^{-11}$ | $-3.209\,829\,829\,373\,991\,210^{-8}$ |
| $\Sigma_q$ | $-0.004\,750\,750\,717\,870\,1073$ | $-0.082\,995\,995\,760\,387\,631$ | $0.004\,959\,959\,517\,708\,477$ |

## 6. Results

### 6.1. Summary of asymptotic expansions

Results obtained so far may be summarized as follows: the resistance $R_{M\times N}(r, s)$ is given by (3), with $R_{M\times N}(\rho)$ expanded as

$$R_{M\times N}(\rho) = \frac{2}{\pi}\log N + \sqrt{\rho}\left(\frac{M}{N} - \frac{1}{2}\right) + \frac{1}{\pi}\left[2\gamma - 1 + 2\log\left(\frac{2}{\pi}\right)\right.$$
$$\left. - \log(1+\rho) + \frac{\rho-1}{\sqrt{\rho}}\tan^{-1}\sqrt{\rho}\right] + L_N(\rho) + \sum_{q=1}^{N/2}\Delta_{q,M,N}(\rho), \tag{39}$$

where $\gamma = 0.577\,215\,664\,901\,53\ldots$ is Euler's constant, $L_N(\rho)$ is given by (28) and $\Delta_{q,M,N}(\rho)$ is given by (37).

As $N \to \infty$ with $\lambda = M/N$ fixed, (39) can be written as

$$R_{M\times N}(\rho) = \frac{2}{\pi}\log N + C(\lambda, \rho) + \sum_{i=1}^{\infty}\frac{b_{2i}(\lambda, \rho)}{N^{2i}}, \tag{40}$$

where

$$C(\lambda, \rho) = \sqrt{\rho}\left(\lambda - \frac{1}{2}\right) + \frac{1}{\pi}\left[2\gamma - 1 + 2\log\left(\frac{2}{\pi}\right)\right.$$
$$\left. - \log(1+\rho) + \frac{\rho-1}{\sqrt{\rho}}\tan^{-1}\sqrt{\rho}\right] + \sum_{q=1}^{\infty}\Delta_{q,0}(\lambda, \rho), \tag{41}$$

$$b_{2i}(\lambda, \rho) = -\left(\frac{2B_{2i}\pi^{2i-1}}{(2i)!}\right)f_\rho^{(2i-1)}(0) + \sum_{q=1}^{\infty}\Delta_{q,2i}(\lambda, \rho).$$

Here, the Bernoulli numbers are $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$ ([9] equation (5.8.8)). The function $f_\rho(x)$ is defined by (23) and its first few derivatives are given in (30). Equation (37) gives an expansion of $\Delta_{q,M,N}(\rho)$ in inverse powers of $N^2$ correct to $O(1/N^4)$, and the coefficients decay exponentially with $q$ so that accurate results may be obtained using only the first few terms of the sum. This is illustrated in table 1 in the case $\lambda = \rho = 1$.

### 6.2. The case $M = N$, $r = s = 1$

For an $N \times N$ network with $r = s = 1$ we have $\lambda = \rho = 1$. From (3) and (40) we obtain

$$R_{N\times N}(1, 1) = 2R_{N\times N}(1)$$
$$= \frac{4}{\pi}\log N + c_0 + \frac{c_2}{N^2} + \frac{c_4}{N^4} + O\left(\frac{1}{N^6}\right), \tag{42}$$

where

$$
\begin{aligned}
c_0 &= 2C(1,1) + 2\sum_{q=1}^{\infty} \Delta_{q,0}(1,1) \\
&= 1 + \frac{2}{\pi}\left[2\gamma - 1 + \log\left(\frac{2}{\pi^2}\right)\right] + \frac{4}{\pi}\sum_{q=1}^{\infty}\left(\frac{\tanh(\pi q)-1}{q}\right) \\
&= (0.082\,069\,879\,627\,328\ldots) - (0.004\,750\,985\,717\,870\,0465\ldots) \\
&= 0.077\,318\,893\,909\,458\ldots, \\
c_2 &= -2\pi B_2 f_1^{(1)}(0) + 2\sum_{q=1}^{\infty}\Delta_{q,2}(1,1) \\
&= 0.266\,070\,441\,638\,478\ldots, \\
c_4 &= -\frac{\pi^3 B_4}{6}f_1^{(3)}0) + 2\sum_{q=1}^{\infty}\Delta_{q,4}(1,1) \\
&= -0.534\,779\,473\,843\,066\ldots,
\end{aligned}
\tag{43}
$$

where we have used the data in table 1. This reproduces numerical values of the coefficient $c_0$ determined from a differential approximant analysis [6] of the first 29 values of $R_{N\times N}(1,1)$ together with a Neville table analysis [7]. Note that the correction to the dominant contribution in $c_0$ is not negligible. We have further extended the Neville table analysis of [7] to the next two coefficients, and obtained results in agreement with the theoretical values of $c_2$ and $c_4$.

Finally, the asymptotic expansion (42) is to be compared to that of the resistance between nodes $(0,0)$ and $(N-1, N-1)$ in an infinite square lattice [4],

$$
R_{N\times N,\infty}(1,1) = \frac{1}{\pi}[\log N + \gamma + 2\log 2] + \cdots.
\tag{44}
$$

## Acknowledgments

## Appendix

In this appendix we evaluate $S_{q,M,N}(\rho)$ given by the summation (8) using the Euler–Maclaurin sum formula ([9] equation (5.8.18))

$$
\begin{aligned}
\sum_{k=0}^{r-1} g_{k+\frac{1}{2}} = \frac{1}{h}\int_{x_0}^{x_r} g(x)\,\mathrm{d}x \\
-\sum_{i=1}^{m}\frac{(1-2^{1-2i})B_{2i}h^{2i-1}}{(2i)!}[g^{(2i-1)}(x_r) - g^{(2i-1)}(x_0)] + E_m(\xi_m),
\end{aligned}
\tag{A.1}
$$

where $g(x)$ is such that $g_i = g(x_0 + ih)$, the integer $r$ is finite, and

$$
E_m(\xi_m) = -r\frac{(1-2^{-1-2m})B_{2m+2}h^{2m+2}}{(2m+2)!}g^{(2m+2)}(\xi_m), \qquad x_0 < \xi_m < x_r,
$$

where $B_{2m}$ are Bernoulli numbers.

Using (A.1) with $x_0 = 0$, $x_r = \pi/2$, $h = \pi/M$, $r = M/2$,

$$g(x) = \frac{1}{\rho \sin^2 A_q + \sin^2 x} \tag{A.2}$$

and noting that the odd derivatives vanish at the endpoints, we obtain

$$S_{q,M,N}(\rho) = \frac{1}{\pi} \int_0^{\pi/2} g(x)\,\mathrm{d}x + E_m(q, M, N, \xi_m)$$

$$= \frac{1}{2\sqrt{\rho}\sin A_q \sqrt{1 + \rho \sin^2 A_q}} + E_m(q, M, N, \xi_m). \tag{A.3}$$

The comparison of (A.3) with (14) indicates that the dominant contribution of $S_{q,M,N}(\rho)$ is precisely (14) with $\tanh(\pi y^*)$ replaced by 1, a result we quoted earlier. It also identifies the error term to be

$$E_m(q, M, N, \xi_m) = \frac{\tanh(\pi y^*) - 1}{2\sqrt{\rho}\sin A_q \sqrt{1 + \rho \sin^2 A_q}}, \tag{A.4}$$

a result which cannot be deduced from the Euler–Maclaurin formula. We point out that since the denominator of (A.2) can be very small for $q$ and $x$ small, $E_m(q, M, N, \xi_m)$ does not necessarily vanish even in the limit of $m \to \infty$.

*Note added in proof.* Equation (9) may also be derived from equation (A.3) of [10]. We thank Professor R Kenna for drawing our attention to this reference.

## References

[1] Kirchhoff G 1847 *Ann. Phys. Chem.* **72** 497–508
[2] van der Pol B 1959 *Lectures in Applied Mathematics* vol 1 ed M Kac (London: Interscience) pp 237–57
[3] Doyle P G and Snell J L 1984 *Random Walks and Electric Networks* (*The Carus Mathematical Monograph, Series 22*) (Washington, DC: The Mathematical Association of America) pp 83–149 (arXiv:math.PR/0001057)
[4] Cserti J 2000 *Am. J. Phys.* **68** 896–906
[5] Domany E and Kinzel W 1984 *Phys. Rev. Lett.* **53** 311–4
[6] Guttmann A J 1989 *Phase Transitions and Critical Phenomena* vol 13 ed C Domb and J L Lebowitz (New York: Academic) pp 1–229
[7] Essam J W, TanlaKishani D and Bhatti F M unpublished report available at the website http://personal.rhul.ac.uk/uhah/101
[8] Wu F Y 2004 *J. Phys. A: Math. Gen.* **37** 6653–73
[9] Hildebrand F B 1956 *Introduction to Numerical Analysis* (Bombay-Dehli: Tata McGraw-Hill)
[10] Janke W and Kenna R 2002 *Phys. Rev.* B **26** 064110